

ISOMORPHISMS BETWEEN EIGENSPACES OF SLOW AND FAST TRANSFER OPERATORS

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ABSTRACT. For any Hecke triangle surface $\Gamma \backslash \mathbb{H}$ of finite or infinite area and any finite-dimensional unitary representation χ of the Hecke triangle group Γ there had been constructed two families of Ruelle-like transfer operators parametrized by \mathbb{C} . The eigenfunctions with eigenvalue 1 of the transfer operators $\mathcal{L}_s^{\text{fast}}$ of the one family determine the zeros of the Selberg zeta function for (Γ, χ) . Further, if $\Gamma \backslash \mathbb{H}$ is cofinite and χ is trivial, the eigenfunctions with eigenvalue 1 of a certain regularity of the transfer operators $\mathcal{L}_s^{\text{slow}}$ in the other family characterize the Maass cusp forms for Γ .

In this article we characterize this eigenspace of $\mathcal{L}_s^{\text{fast}}$ as an eigenspace with eigenvalue 1 of $\mathcal{L}_s^{\text{slow}}$, and vice versa. This solves a conjecture by the second author and M. Möller.

1. INTRODUCTION

Let $\mathbb{H} = \text{PSL}_2(\mathbb{R}) / \text{PSO}(2)$ denote the hyperbolic plane, let Γ be a Fuchsian group, and let $\chi: \Gamma \rightarrow \text{U}(V)$ be a unitary representation of Γ on a finite-dimensional vector space V . The relation between the geometric and the spectral properties of $X := \Gamma \backslash \mathbb{H}$ (e.g., volume, periodic geodesics, etc., among the geometric objects, and eigenvalues, resonances, (Γ, χ) -automorphic functions, etc., among the spectral entities) is an important subject with a long, rich history. Among the various methods used in the study of this relation, one is the development of transfer operator techniques.

The articles [16, 9, 19, 13, 18, 17, 21, 20, 15, 1] document part of a program to systematically develop dual ‘slow/fast’ transfer operator approaches to automorphic functions, resonances and Selberg zeta functions (cf. Figure 1) for a certain class of (cofinite and non-cofinite) Fuchsian groups Γ with cusps.

In Figure 1, all entities depend on $X = \Gamma \backslash \mathbb{H}$. Here, $Z = Z_{\Gamma, \chi}$ denotes the Selberg zeta function of (Γ, χ) , and MCF_s the space of Maass cusp forms for Γ with spectral parameter s . Further, ‘slow’ refers to that each point of the discrete dynamical system used in the definition of the transfer operator has finitely many preimages only, or equivalently, that the symbolic dynamics arising from the discretization of the geodesic flow on X uses a finite alphabet only (see [16, 19]). Hence, ‘slow’ transfer operators involve finite sums only. In contrast, ‘fast’ means that points

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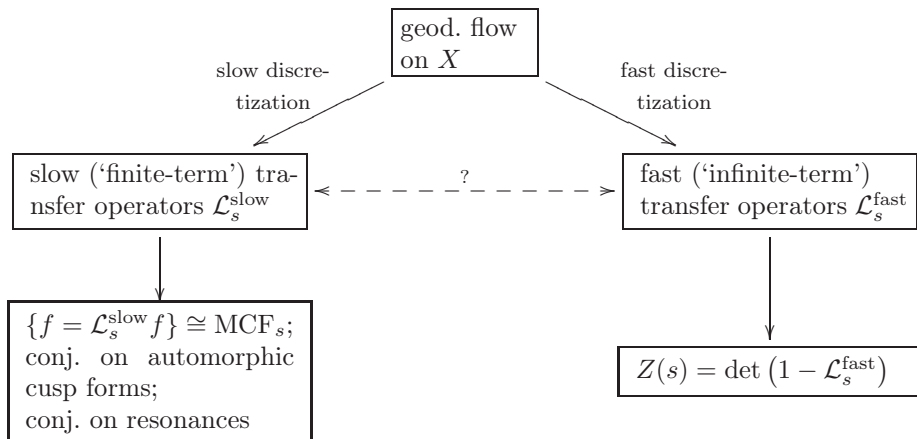


FIGURE 1. Dual transfer operator approaches

with infinitely but countably many preimages occur, and hence the associated ‘fast’ transfer operators involve infinite sums.

We refer to Section 3 for examples of these transfer operators. Further, we refer to the already mentioned articles and the references therein for a more comprehensive exposition of such transfer operator approaches, their history and their relation to mathematical quantum chaos and other areas, and remain here rather brief.

If χ is the trivial one-dimensional representation and Γ is a lattice (that is admissible for these techniques) then the slow transfer operators provide a dynamical characterization of the Maass cusp forms for Γ [17]. More precisely, for $s \in \mathbb{C}$, $\text{Re } s \in (0, 1)$, the Maass cusp forms with spectral parameter s are isomorphic to the eigenfunctions of the transfer operator $\mathcal{L}_s^{\text{slow}}$ with eigenvalue 1 of sufficient regularity (‘period functions’). The proof of the isomorphism between Maass cusp forms and these period functions takes advantage of the characterization of Maass cusp forms in parabolic cohomology as provided by [3]. Both, [17] and [3] do not rely on Selberg theory.

In [13, 21, 20, 15], the construction of fast transfer operators $\mathcal{L}_s^{\text{fast}}$ is conducted with all details only for (cofinite and non-cofinite) Hecke triangle groups. Anyhow, the structure of these constructions clearly applies to a wider class of Fuchsian groups. It shows that the Fredholm determinant of the family of fast transfer operators $\mathcal{L}_s^{\text{fast}}$ represents the Selberg zeta function Z_Γ of Γ , and hence the zeros of Z_Γ are determined by the eigenfunctions of $\mathcal{L}_s^{\text{fast}}$ with eigenvalue 1 [13, 15]. Also this proof is independent of Selberg theory.

As is well-known, Selberg theory shows a relation between (some of) the zeros of Z_Γ and the spectral parameters of the Maass cusp forms for Γ . Hence it provides a link (on the spectral level) between the two bottom objects in Figure 1. It is natural to ask if such a link can be proven as well within the framework of transfer operators without using Selberg theory and if it can be seen even on the level of

the eigenfunctions of the transfer operators (the dotted ‘?’-arrow) as conjectured in [13, 20, 15].

If χ is non-trivial or Γ is non-cofinite then analogous constructions of transfer operators apply. Also in these setups, the fast transfer operators determine the Selberg zeta functions $Z_{\Gamma, \chi}$ for (Γ, χ) [20, 15]. The role of the eigenfunctions of the slow transfer operators is not yet fully understood. Conjecturally they give rise to period functions for vector-valued automorphic cusp forms or, for non-cofinite Fuchsian groups, determine the image of the residue operators at resonances [15]. By analogy, a close relation between the eigenfunctions with eigenvalue 1 of $\mathcal{L}_s^{\text{slow}}$ and $\mathcal{L}_s^{\text{fast}}$ is expected [20, 15].

We show that these eigenfunction spaces are indeed isomorphic for any Hecke triangle group.

Theorem A. *Let Γ be a (cofinite or non-cofinite) Hecke triangle group and χ a finite-dimensional unitary representation of Γ , and let $\text{Re } s > 0$. Suppose that $\mathcal{L}_s^{\text{slow}}$ and $\mathcal{L}_s^{\text{fast}}$ are the associated families of slow respectively fast transfer operators. Then the eigenfunctions with eigenvalue 1 of $\mathcal{L}_s^{\text{fast}}$ are isomorphic to the real-analytic eigenfunctions with eigenvalue 1 of $\mathcal{L}_s^{\text{slow}}$ that satisfy a certain growth restriction.*

The isomorphism in Theorem A is explicit and constructive. Moreover, if Γ is a lattice and χ is the trivial representation then the period functions (i.e., those eigenfunctions of $\mathcal{L}_s^{\text{slow}}$ that are isomorphic to the Maass cusp forms for Γ with spectral parameter s) can be characterized as a certain subspace of the eigenfunctions of $\mathcal{L}_s^{\text{fast}}$. More generally, additional conditions of a certain type on the eigenfunctions of $\mathcal{L}_s^{\text{slow}}$ translate to essentially the same conditions on the eigenfunctions of $\mathcal{L}_s^{\text{fast}}$. We refer to Theorems 3.4, 3.14 and 3.15 below for more details.

Neither the proof of Theorem A nor the characterization of the subspace of eigenfunctions of $\mathcal{L}_s^{\text{fast}}$ that corresponds to period functions—and hence Maass cusp forms—uses Selberg theory. However, these results do not allow us yet to classify the zeros of the Selberg zeta function within this transfer operator framework. For the case that $\Gamma = \text{PSL}_2(\mathbb{Z})$ and χ is the trivial representation more is known due to the combination of [12, 6, 5, 2, 8]. We comment on it in more details in Section 4 below.

The restriction to Hecke triangle groups is a consequence of the fact that in [13, 20, 15] the details are provided for these Fuchsian groups only. However, due to the systematic structure of all constructions and proofs in [13, 20, 15] and in this article, it is obvious that Theorem A easily generalizes to a wider class of cofinite and non-cofinite Fuchsian groups. We briefly elaborate on this in Section 4 below.

The restriction to Hecke triangle groups allows us to actually prove a stronger statement than Theorem A. Each Hecke triangle group commutes with a certain element $Q \in \text{PGL}_2(\mathbb{R})$ of order 2, which acts as an orientation-reversing Riemannian isometry on \mathbb{H} . This exterior symmetry is compatible with the transfer operators, and hence induces their splitting into the odd parts $\mathcal{L}_s^{\text{slow}, -}$ and $\mathcal{L}_s^{\text{fast}, -}$ as well as the even parts $\mathcal{L}_s^{\text{slow}, +}$ and $\mathcal{L}_s^{\text{fast}, +}$, respectively. If Γ is cofinite, χ is the trivial representation and $\text{Re } s \in (0, 1)$ then the sufficiently regular eigenfunctions with eigenvalue

1 of $\mathcal{L}_s^{\text{slow},+}$ resp. of $\mathcal{L}_s^{\text{slow},-}$ (equivalently the eigenfunctions with eigenvalue 1 of $\mathcal{L}_s^{\text{slow}}$ that are invariant resp. anti-invariant under the action of Q) are isomorphic to the even resp. odd Maass cusp forms for Γ [13, 21]. The Selberg-type zeta functions for the even resp. odd spectrum of Γ equal the Fredholm determinant of the transfer operator families $\mathcal{L}_s^{\text{fast},\pm}$ [21].

Instead of Theorem A we show its strengthened version that considers separately the odd and even transfer operators.

Theorem B. *Let Γ be a (cofinite or non-cofinite) Hecke triangle group, χ a finite-dimensional unitary representation of Γ , and $\text{Re } s > 0$, and suppose that $\mathcal{L}_s^{\text{slow},\pm}$ and $\mathcal{L}_s^{\text{fast},\pm}$ are the associated families of slow/fast even/odd transfer operators. Then the real-analytic eigenfunctions with eigenvalue 1 of $\mathcal{L}_s^{\text{slow},+}$ (resp. $\mathcal{L}_s^{\text{slow},-}$) that satisfy a certain growth condition are isomorphic to the eigenfunctions with eigenvalue 1 of $\mathcal{L}_s^{\text{fast},+}$ (resp. $\mathcal{L}_s^{\text{fast},-}$).*

The same comments as for Theorem A apply to Theorem B. In particular, the isomorphism in Theorem B is explicit and constructive, and certain additional conditions on eigenfunctions can be accommodated. Therefore, even and odd Maass cusp forms can be characterized as certain eigenfunctions of $\mathcal{L}_s^{\text{fast},\pm}$, respectively. Again we refer to Theorems 3.4, 3.14 and 3.15 below for precise statements. For the modular group $\text{PSL}_2(\mathbb{Z})$ and χ being the trivial representation, Theorem B has already been proven in [12] and [6] (independently of each other). We comment on this in Remark 3.5 below.

Patterson [14] proposed a cohomological framework for the divisors of Selberg zeta functions. If Γ is a lattice and χ is the trivial representation then—as mentioned above—certain eigenspaces of $\mathcal{L}_s^{\text{slow}}$ for the eigenvalue 1 are isomorphic to parabolic 1-cohomology spaces, and hence Theorems A and B support Patterson’s conjecture. For the case that Γ is not a lattice or χ is non-trivial, Theorems A and B support the conjectures on the significance of the eigenfunctions of $\mathcal{L}_s^{\text{slow}}$. We discuss this further in Section 4 below.

In Section 2 we provide the necessary background on Hecke triangle groups and transfer operators. In Section 3 we prove Theorems A and B, and in the final Section 4 we briefly comment on the underlying structure of the isomorphism maps for Theorems A and B, and the possibility for their generalizations.

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2. PRELIMINARIES

2.1. The hyperbolic plane. As model for the hyperbolic plane we use the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$$

endowed with the well-known hyperbolic Riemannian metric given by the line element

$$ds^2 = \frac{dzd\bar{z}}{(\text{Im } z)^2}.$$

We identify its geodesic boundary with $P^1(\mathbb{R}) \cong \mathbb{R} \cup \{\infty\}$. The action of the group of Riemannian isometries on \mathbb{H} extends continuously to $P^1(\mathbb{R})$.

This group of isometries is isomorphic to

$$G := \mathrm{PGL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})/(\mathbb{R}^\times \cdot \mathrm{id}),$$

its subgroup of orientation-preserving Riemannian isometries is

$$\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm \mathrm{id}\}.$$

The action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H} \cup P^1(\mathbb{R})$ is then given by fractional linear transformations, i. e., for $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ and $z \in \mathbb{H} \cup \mathbb{R}$ we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \begin{cases} \frac{az+b}{cz+d} & \text{for } cz+d \neq 0 \\ \infty & \text{for } cz+d = 0 \end{cases} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \infty = \begin{cases} \frac{a}{c} & \text{for } c \neq 0 \\ \infty & \text{for } c = 0. \end{cases}$$

2.2. Hecke triangle groups. The Hecke triangle group Γ_ℓ with parameter $\ell > 0$ is the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by the two elements

$$(1) \quad S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad T_\ell := \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix}.$$

It is Fuchsian if and only if $\ell \geq 2$ or $\ell = 2 \cos \frac{\pi}{q}$ with $q \in \mathbb{N}_{\geq 3}$. In the following, the expression ‘Hecke triangle group’ always refers to a Fuchsian Hecke triangle group, and we refer to the spaces $X_\ell = \Gamma_\ell \backslash \mathbb{H}$ as *Hecke triangle surfaces*.

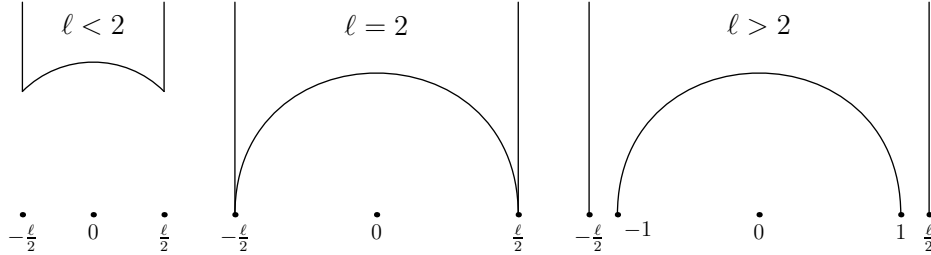


FIGURE 2. Fundamental domain for Γ_ℓ .

A fundamental domain for Γ_ℓ is given by (see Figure 2)

$$\mathcal{F}_\ell := \{z \in \mathbb{H} \mid |z| > 1, |\operatorname{Re} z| < \ell/2\}.$$

The side-pairings for \mathcal{F}_ℓ are provided by the generators (1): the vertical sides $\{\operatorname{Re} z = -\ell/2\}$ and $\{\operatorname{Re} z = \ell/2\}$ are identified via T_ℓ , and the bottom sides $\{|z| = 1, \operatorname{Re} z \leq 0\}$ and $\{|z| = 1, \operatorname{Re} z \geq 0\}$ via S .

Among the Hecke triangle groups only those with parameters $\ell \leq 2$ are lattices. For $\ell = \ell(q) = 2 \cos \frac{\pi}{q}$ with $q \in \mathbb{N}_{\geq 3}$, X_ℓ has one cusp (represented by ∞) and two elliptic points. In the special case $q = 3$, thus $\ell(q) = 1$, the Hecke triangle group Γ_1 is the modular group $\mathrm{PSL}_2(\mathbb{Z})$. The Hecke triangle group Γ_2 is commonly known as the Theta group. It is conjugate to the projective version of $\Gamma_0(2)$. The associated Hecke triangle surface X_2 has two cusps (represented by ∞ and $\ell/2$) and one elliptic point. The groups Γ_ℓ for $\ell > 2$ are non-cofinite, the orbifold X_ℓ

has one funnel (represented by the subset $[-\ell/2, -1] \cup (1, \ell/2)$ of \mathbb{R}), one cusp (represented by ∞) and one elliptic point. The Hecke triangle groups Γ_ℓ with $\ell \in \{\ell(3), \ell(4), \ell(6), 2\}$ are the only arithmetic ones.

Thus, Hecke triangle groups form a one-parameter family consisting of cofinite and non-cofinite as well as arithmetic and non-arithmetic Fuchsian groups, and it contains the well-investigated modular group.

2.3. Representations and actions. Let χ be a finite-dimensional unitary representation of $G = \mathrm{PGL}_2(\mathbb{R})$ on the complex vector space V .

Let $s \in \mathbb{C}$ and $g \in \mathrm{PSL}_2(\mathbb{R})$. For a function $f: I \rightarrow V$ on a subset I of \mathbb{R} we define

$$\alpha_s(g^{-1})f(x) := |g'(x)|^s \chi(g)f(g.x),$$

whenever it makes sense.

In order to define a highly regular (continuous respectively holomorphic) continuation of this action to all of G and to functions defined on subsets of \mathbb{C} we define the action of $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$ on $\mathbb{C} \setminus \{-c/d\}$ (or even the whole Riemann sphere) by fractional linear transformation:

$$g.z := \frac{az + b}{cz + d}.$$

Suppose that $d \neq 0$. For $x \in \mathbb{R} \setminus \{-c/d\}$ we then have

$$(2) \quad |g'(x)|^s = (|ad - bc| \cdot (cx + d)^{-2})^s = |ad - bc|^s |cx + d|^{-2s}.$$

Among the real numbers we use here (2) for $cx + d > 0$ only.

We use the principal branch for the complex logarithm (i.e., with the cut plane $\mathbb{C} \setminus (-\infty, 0]$). For the holomorphic continuation of (2) we then have two possibilities depending on whether we extend the first or the second expression.

From the point of view of transfer operators, the first expression is the more natural one. It extends by

$$j_s^{(1)}(g, z) := (|ad - bc| \cdot (cz + d)^{-2})^s$$

holomorphically to

$$C_{(1)} := \{z \in \mathbb{C} \mid \operatorname{Re} z > -c/d\}.$$

For other approaches to and applications of period functions the second expression is sometimes used. It extends by

$$j_s^{(2)}(g, z) := |ad - bc|^s |cz + d|^{-2s}$$

holomorphically to

$$C_{(2)} := \mathbb{C} \setminus (-\infty, -c/d].$$

Obviously, on $C_{(1)}$ both extensions are identical. For $j \in \{1, 2\}$ and a function $f: W \rightarrow V$ on some subset $W \subseteq C_{(j)}$ we set

$$\alpha_s^{(j)}(g^{-1})f(z) := j_s^{(j)}(g, z)\chi(g)f(g.z).$$

We write just α_s for generic results or if the choice is understood. The statements and proofs of Theorems A and B do not depend on the choice. It only affects an intermediate result on the maximal domain of holomorphy for certain functions, see Propositions 3.7 and 3.8 below.

2.4. Meromorphic continuations. Let $h \in \mathrm{PSL}_2(\mathbb{R})$ be a parabolic element. For $\mathrm{Re} s > \frac{1}{2}$, the infinite sum

$$(3) \quad \mathcal{N}_s := \sum_{k=1}^{\infty} \alpha_s(h^k)$$

defines an operator between various spaces of functions, for examples see below or [13]. Taking advantage of the well-known properties of the Lerch zeta function, either in the form

$$\zeta(s, a, w) = \sum_{n=0}^{\infty} \frac{e^{2\pi i n a}}{((n+w)^2)^{s/2}}$$

if we use $\alpha_s^{(1)}$ for α_s , or in the form

$$\zeta(s, a, w) = \sum_{n=0}^{\infty} \frac{e^{2\pi i n a}}{(n+w)^s}$$

if we use $\alpha_s^{(2)}$ for α_s , the map

$$s \mapsto \mathcal{N}_s$$

extends meromorphically to all of \mathbb{C} . All its poles are simple and contained in $\frac{1}{2} - \frac{1}{2}\mathbb{N}_0$. The existence of poles intimately depends on the degree of singularity of the representation χ (cf. [15]).

Throughout, for any operator of the form (3), we denote its meromorphic continuation by \mathcal{N}_s as well (more precisely, with the same symbol as the initial operator for $\mathrm{Re} s > \frac{1}{2}$). Further, to simplify notation, we use \mathcal{N}_s to denote any operator which acts by (3), the specific spaces on which we consider its action are always understood. Finally, whenever we use an expression that involves \mathcal{N}_s and ‘all’ $s \in \mathbb{C}$ then it is understood that we exclude the poles.

2.5. Transfer operators. For a given discrete dynamical system $F: D \rightarrow D$, the associated transfer operator $\mathcal{L}_{\varphi, w}$ with potential $\varphi: D \rightarrow \mathbb{C}$ and weight function w is defined by

$$\mathcal{L}f(x) := \sum_{y \in F^{-1}(x)} w(y) e^{\varphi(y)} f(y),$$

acting on an appropriate space of functions f (to be adapted to the discrete dynamical system and the applications under consideration).

The transfer operators we consider in this article are more specific. They have been developed in [13, 21, 20, 15]. We survey their common properties that are important for the proofs of Theorems A and B. We refer to the original articles as well as to the following sections for more details.

Let Γ denote a Hecke triangle group and let $\tilde{\Gamma} \subseteq \mathrm{PGL}_2(\mathbb{R})$ be its underlying triangle group. The discrete dynamical systems (D, F) that we use in the transfer operator for Γ arise from a discretization and symbolic dynamics for the geodesic flow on $X = \Gamma \backslash \mathbb{H}$ (or rather $\tilde{\Gamma} \backslash \mathbb{H}$). The set D is a family of real intervals D_κ , $\kappa \in K$ for some (finite or countable) index set K , and the map F is determined by a family

$$(4) \quad F_k := F|_{D_k}: D_k \rightarrow F_k(D_k)$$

of diffeomorphisms that are identical to the action of certain elements in $\tilde{\Gamma}$. The potentials we are interested in are $\varphi_s(y) = -s \log |F'(y)|$ for $s \in \mathbb{C}$. The weight function depends on the finite-dimensional unitary representation (V, χ) and whether we intend to investigate the odd ('-') or the even ('+') spectrum.

For the parameter $s \in C$, we denote the even transfer operator by \mathcal{L}_s^+ and the odd transfer operator by \mathcal{L}_s^- . Since we consider the representation (V, χ) to be fixed throughout, we omit it from the notation.

For a subset $I \subseteq \mathbb{R}$ let

$$\text{Fct}(I; V) := \{f: I \rightarrow V\}$$

denote the space of functions $I \rightarrow V$. Formally, any arising transfer operator \mathcal{L}_s^\pm is represented by a matrix

$$\mathcal{L}_s^\pm = \left(\mathcal{L}_{s,a,b}^\pm \right)_{a,b \in \mathcal{A}}$$

for a finite index set \mathcal{A} and acts on function vectors

$$f = (f_a)_{a \in \mathcal{A}}$$

where, for each $a \in \mathcal{A}$,

$$f_a \in \text{Fct}(I_a; V)$$

for some interval $I_a \subseteq \mathbb{R}$. The intervals are closely related to the sets $F_k(D_k)$ in (4). Further, for any $a, b \in \mathcal{A}$ there is a (finite or countable) index set $C_{a,b}$ and for each $c \in C_{a,b}$ an element $g_c^{(a,b)} \in \tilde{\Gamma}$ such that

$$(5) \quad \mathcal{L}_{s,a,b}^\pm = \sum_{c \in C_{a,b}} w(g_c^{(a,b)}) \alpha_s(g_c^{(a,b)}).$$

The weight function is given by $w: G \rightarrow \{\pm 1\}$,

$$w(g) := \begin{cases} 1 & \text{for even ('+') transfer operators} \\ \text{sign}(\det(g)) & \text{for odd ('-') transfer operators.} \end{cases}$$

Recall that the action α_s depends on the representation χ . Moreover, for any $a, b \in \mathcal{A}$ and $c \in C_{a,b}$ we have

$$\left(g_c^{(a,b)} \right)^{-1} \cdot I_a \subseteq I_b.$$

While this latter property ensures well-definedness for each single summand in (5), there might be a convergence problem for the potentially infinite sums.

As indicated in Figure 1, the discretizations and symbolic dynamics we use here come in pairs: a slow version and a fast version. The fast version is deduced from the slow one by a certain induction process on certain parabolic elements; we refer to the already mentioned articles for details. Therefore, also the odd and even transfer operators come in pairs: the slow odd and even transfer operators $\mathcal{L}_s^{\text{slow}, \pm}$ for which all index sets $C_{a,b}$ in (5) are finite, and the fast odd and even transfer operators which also have infinite terms.

2.5.1. *Slow transfer operators.* For the odd and even slow transfer operators $\mathcal{L}_s^{\text{slow}, \pm}$ for Hecke triangle groups Γ , the index set \mathcal{A} consists of a single element only. For this reason we omit it from the notation. The index set C is finite, its precise number of elements depends on Γ . Thus, the slow transfer operators indeed act on $\text{Fct}(I; V)$. For our applications we consider them to act on the real-analytic functions $C^\omega(I; V)$ and we are interested in the space (‘real-analytic odd/even slow eigenfunctions for the parameter s ’)

$$\text{SEF}_s^{\omega, \pm} := \{f \in C^\omega(I; V) \mid \mathcal{L}_s^{\text{slow}, \pm} f = f\},$$

more precisely, in a certain subspace $\text{SEF}_s^{\omega, \text{as}, \pm}$ of functions satisfying certain growth restrictions as well as a certain subspace $\text{SEF}_s^{\omega, \text{dec}, \pm}$ of functions obeying certain decay properties. These properties depend on the specific Hecke triangle group, for which reason we refer to Sections 3.1-3.3 for the definitions.

Theorem 2.1 ([13, 21, 15]). *Let Γ be a lattice, χ be the trivial representation, and $\text{Re } s \in (0, 1)$. Then $\text{SEF}_s^{\omega, \text{dec}, \pm}$ is isomorphic to the space of odd (if ‘−’) respectively even (if ‘+’) Maass cusp forms with spectral parameter s for Γ .*

2.5.2. *Fast transfer operators.* For any fast transfer operator, at least one of the index sets $C_{a,b}$ in (5) is infinite and hence causes a convergence problem. However, the structure of the infinite sums is controlled and allows for a uniform treatment.

The purpose of the fast transfer operators is to represent Selberg zeta functions as Fredholm determinants. In order to fulfill this purpose, we consider the fast transfer operator on a certain Banach space on which it acts as a nuclear operator of order 0.

More precisely, for $a \in \mathcal{A}$ we fix an open connected complex neighborhood \mathcal{E}_a (in the Riemann sphere) of the closure \bar{I}_a of the real interval I_a such that for all $b \in \mathcal{A}$ and all $c \in C_{a,b}$ we have

$$\left(g_c^{(a,b)}\right)^{-1} \cdot \bar{\mathcal{E}}_a \subseteq \mathcal{E}_b.$$

Define

$$B(\mathcal{E}_a) := \{\psi: \bar{\mathcal{E}}_a \rightarrow V \text{ continuous} \mid \psi|_{\mathcal{E}_a} \text{ holomorphic}\}.$$

Endowed with the supremum norm, $B(\mathcal{E}_a)$ is a Banach space. Further set

$$B(\mathcal{E}) := \bigoplus_{a \in \mathcal{A}} B(\mathcal{E}_a)$$

to be the direct product of these Banach spaces.

If $(\mathcal{E}'_a)_{a \in \mathcal{A}}$ is also a family of complex sets with these inclusion properties then we define

$$(\mathcal{E}'_a)_{a \in \mathcal{A}} \leq (\mathcal{E}_a)_{a \in \mathcal{A}}$$

if and only if

$$\mathcal{E}'_a \subseteq \mathcal{E}_a \quad \text{for all } a \in \mathcal{A}.$$

Let

$$\mathcal{B} := \mathcal{B}(I) := \bigoplus_{a \in \mathcal{A}} \mathcal{B}(I_a) := \text{proj} \lim_{a \in \mathcal{A}} \bigoplus_{a \in \mathcal{A}} B(\mathcal{E}_a)$$

denote the projective limit of the Banach spaces.

- Theorem 2.2** ([21, 20, 15]). (i) For $\operatorname{Re} s > \frac{1}{2}$, the transfer operators $\mathcal{L}_s^{\text{fast}, \pm}$ act on \mathcal{B} as a nuclear operator of order 0.
- (ii) The map $s \mapsto \mathcal{L}_s^{\text{fast}, \pm}$ extends to a meromorphic function on \mathbb{C} with values in nuclear operators of order 0 on \mathcal{B} . The possible poles are all simple and contained in $\frac{1}{2}(1 - \mathbb{N}_0)$.
- (iii) The Selberg zeta function Z for (Γ, χ) equals the Fredholm determinant

$$Z(s) = \det(1 - \mathcal{L}_s^{\text{fast}, +}) \det(1 - \mathcal{L}_s^{\text{fast}, -}).$$

- (iv) If Γ is a lattice with a single cusp and χ is the trivial representation then $\det(1 - \mathcal{L}_s^{\text{fast}, \pm})$ equals the Selberg-type zeta function Z_{\pm} for the odd (if ‘-’) respectively the even (if ‘+’) spectrum:

$$Z_{\pm}(s) = \det(1 - \mathcal{L}_s^{\text{fast}, \pm}).$$

For $s \in \mathbb{C}$ we define (‘odd/even fast eigenfunctions for the parameter s ’)

$$\text{FEF}_s^{\pm} := \{f \in \mathcal{B} \mid f = \mathcal{L}_s^{\text{fast}, \pm} f\}.$$

The elements of FEF_s^{\pm} determine the zeros of Z_{\pm} respectively of Z .

2.6. Convention. For any $x_0 \in \mathbb{R} \cup \{\pm\infty\}$ and any complex-valued functions $f, g: \mathbb{R} \rightarrow \mathbb{C}$ we use $f(x) = O_{x \rightarrow x_0^+}(g(x))$ for

$$\limsup_{x \searrow x_0} \left| \frac{f(x)}{g(x)} \right| < \infty.$$

Note that, in contrast to other convention, we allow (for simplicity) that g does not need to be positive. We use similar conventions for other symbols from the O -notation.

3. PROOF OF THEOREMS A AND B

We show Theorem B separately for the cofinite Hecke triangle groups with a single cusp, the Theta group, and the non-cofinite Hecke triangle groups. Within these classes, the structure of the groups and transfer operators allows for an easy uniform statement of the maps that provide the claimed isomorphism between the eigenspaces of the slow and fast transfer operators. However, these maps obey a certain uniform structure which indicate that these results apply to other Fuchsian groups as well. In Section 4 below we discuss the possible generalizations in more details.

Throughout let

$$Q := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad J := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3.1. Isomorphism for the Hecke triangle groups Γ_{ℓ} with $\ell < 2$. Let $q \in \mathbb{N}_{\geq 3}$ and set

$$\ell := \ell(q) := 2 \cos \frac{\pi}{q}.$$

For the cofinite Hecke triangle group

$$\Gamma := \Gamma_q := \Gamma_{\ell}$$

with a single cusp we consider the transfer operators developed in [13, 21, 15]. Recall that $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $T := T_q := T_\ell = \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix}$. For $k \in \mathbb{Z}$ let

$$g_{q,k} := ((T_q S)^k S)^{-1}.$$

With

$$s(m, q) := \frac{\sin\left(\frac{m}{q}\pi\right)}{\sin\frac{\pi}{q}}$$

for $m \in \mathbb{Z}$ we then have

$$g_{q,k}^{-1} = \begin{bmatrix} s(k, q) & s(k+1, q) \\ s(k-1, q) & s(k, q) \end{bmatrix}.$$

In order to state the odd and even slow and fast transfer operators for Γ_q let

$$m := \left\lfloor \frac{q-1}{2} \right\rfloor.$$

3.1.1. *The case of odd q .* Suppose that q is odd. The odd resp. even slow transfer operator $\mathcal{L}_{q,s}^{\text{slow},\pm}$ of Γ_q is given by

$$\begin{aligned} \mathcal{L}_{q,s}^{\text{slow},\pm} &= \sum_{k=1}^m \alpha_s(g_{q,-k}) \pm \alpha_s(Qg_{q,-k}) \\ &= (1 \pm \alpha_s(Q)) \sum_{k=1}^m \alpha_s(g_{q,-k}), \end{aligned}$$

acting on $C^\omega((0, 1); V)$. Let

$$\text{SEF}_{q,s}^{\omega,\pm} := \{\varphi \in C_b^\omega((0, 1); V) \mid \varphi = \mathcal{L}_{q,s}^{\text{slow},\pm} \varphi\}$$

denote the space of real-analytic bounded eigenfunctions of $\mathcal{L}_{q,s}^{\text{slow},\pm}$ with eigenvalue 1. Let

$$\text{SEF}_{q,s}^{\omega,\text{as},\pm} := \left\{ \varphi \in \text{SEF}_{q,s}^{\omega,\pm} \mid \exists c \in V : \varphi(x) = \frac{c}{x} + O_{x \rightarrow 0^+}(1) \right\}$$

denote its subspace of functions with a certain controlled growth towards 0, and let $\text{SEF}_{q,s}^{\omega,\text{dec},\pm}$ denote its subspace of functions $\varphi \in \text{SEF}_{q,s}^{\omega,\pm}$ for which the map

$$(6) \quad \begin{cases} \varphi & \text{on } \left(0, \frac{1}{\ell(q)}\right) \\ \mp \alpha_s(J)\varphi & \text{on } \left(-\frac{1}{\ell(q)}, 0\right) \end{cases}$$

extends smoothly (C^∞) to $(-1/\ell(q), 1/\ell(q))$.

Remark 3.1. Note that (6) implies for each $\varphi \in \text{SEF}_{q,s}^{\omega,\text{dec},+}$ we have

$$\lim_{x \rightarrow 0^+} \varphi(x) = 0.$$

Even more, since the limit $\lim_{x \rightarrow 0^+} \varphi'(x)$ exists,

$$\varphi = O_{x \rightarrow 0^+}(x).$$

Proposition 3.2 ([13, 21]). *Suppose that $\text{Re } s \in (0, 1)$ and that χ is the trivial representation. Then $\text{SEF}_{q,s}^{\omega,\text{dec},\pm}$ is isomorphic (as a vector space) to the space of even resp. odd Maass cusp forms for Γ_q with spectral parameter s .*

Remark 3.3. In [13, 21] we consider $\mathcal{L}_{q,s}^{\text{slow},\pm}$ to act on $C^\omega(\mathbb{R}_{>0}; V)$ instead of on $C^\omega((0,1); V)$ and require that

$$(7) \quad \begin{cases} \varphi & \text{on } \mathbb{R}_{>0} \\ -\alpha_s(S)\varphi & \text{on } \mathbb{R}_{<0} \end{cases}$$

extends smoothly to \mathbb{R} instead of asking for (6). However, if $\varphi \in C^\omega(\mathbb{R}_{>0}; V)$ is a eigenfunction with eigenvalue 1 of $\mathcal{L}_{q,s}^{\text{slow},\pm}$ then $\varphi = \pm\alpha_s(Q)\varphi$. Substituting this into (7) and noting that $SQ = J$ shows that (7) is equivalent to (6) up to real-analyticity at 1. However, Proposition 3.7 below shows that each element of $\text{SEF}_{q,s}^{\omega,\pm}$ extends uniquely to an element in $C^\omega(\mathbb{R}_{>0}; V)$. Thus, (6) and (7) are indeed equivalent.

In order to state the fast odd resp. even transfer operator $\mathcal{L}_{q,s}^{\text{fast},\pm}$ of Γ_q we set

$$D_{-1} := \left(0, \frac{1}{\ell(q)}\right) \quad \text{and} \quad D_0 := \left(\frac{1}{\ell(q)}, 1\right)$$

as well as

$$\mathcal{L}_{q,0,s}^{\text{fast}} := \sum_{k=2}^m \alpha_s(g_{q,-k}).$$

For $\text{Re } s > \frac{1}{2}$ we set

$$(8) \quad \mathcal{L}_{q,-1,s}^{\text{fast}} := \sum_{n=1}^{\infty} \alpha_s(g_{q,-1}^n),$$

and have

$$\mathcal{L}_{q,s}^{\text{fast},\pm} = \begin{pmatrix} (1 \pm \alpha_s(Q))\mathcal{L}_{q,0,s}^{\text{fast}} & (1 \pm \alpha_s(Q))\mathcal{L}_{q,-1,s}^{\text{fast}} \\ (1 \pm \alpha_s(Q))\mathcal{L}_{q,0,s}^{\text{fast}} & \pm\alpha_s(Q)\mathcal{L}_{q,-1,s}^{\text{fast}} \end{pmatrix}$$

which acts on the projective Banach space

$$\mathcal{B} := \mathcal{B}(D_0) \oplus \mathcal{B}(D_{-1}).$$

For $\text{Re } s \leq \frac{1}{2}$, $\mathcal{L}_{q,-1,s}^{\text{fast}}$ and $\mathcal{L}_{q,s}^{\text{fast},\pm}$ are given by meromorphic continuation (see Theorem 2.2 or [13, 15]).

For $s \in \mathbb{C}$ let

$$\text{FEF}_{q,s}^\pm := \{f \in \mathcal{B} \mid f = \mathcal{L}_{q,s}^{\text{fast},\pm} f\}$$

denote the space of eigenfunctions in \mathcal{B} of $\mathcal{L}_{q,s}^{\text{fast},\pm}$ with eigenvalue 1. Let $\text{FEF}_{q,s}^{\text{dec},\pm}$ denote its subspace of maps $f = (f_0, f_{-1})^\top \in \text{FEF}_{q,s}^\pm$ for which

$$(9) \quad \begin{cases} (1 + \mathcal{L}_{q,-1,s}^{\text{fast}}) f_{-1} & \text{for } x > 0 \\ \mp\alpha_s(J)(1 + \mathcal{L}_{q,-1,s}^{\text{fast}}) f_{-1} & \text{for } x < 0 \end{cases}$$

extends smoothly to 0 when considered as a function on some punctured neighborhood of 0 in \mathbb{R} .

Theorem 3.4. *Let $s \in \mathbb{C} \setminus \{1/2\}$ such that $\text{Re } s > 0$. Then the spaces $\text{SEF}_{q,s}^{\omega,\text{as},\pm}$ and $\text{FEF}_{q,s}^\pm$ are isomorphic (as vector spaces). The isomorphism is given by*

$$\text{FEF}_{q,s}^\pm \rightarrow \text{SEF}_{q,s}^{\omega,\text{as},\pm}, \quad f = (f_0, f_{-1})^\top \mapsto \varphi,$$

where

$$(10) \quad \varphi|_{D_0} := f_0 \quad \text{and} \quad \varphi|_{D_{-1}} := (1 + \mathcal{L}_{q,-1,s}^{\text{fast}}) f_{-1}.$$

The converse isomorphism is

$$\text{SEF}_{q,s}^{\omega,\text{as},\pm} \rightarrow \text{FEF}_{q,s}^{\pm}, \quad \varphi \mapsto f = (f_0, f_{-1})^\top,$$

where f is determined by

$$(11) \quad f_0 := \varphi|_{D_0} \quad \text{and} \quad f_{-1} := ((1 - \alpha_s(g_{q,-1}))\varphi)|_{D_{-1}}.$$

These isomorphisms induce isomorphisms between $\text{SEF}_{q,s}^{\omega,\text{dec},\pm}$ and $\text{FEF}_{q,s}^{\text{dec},\pm}$.

Remark 3.5. For $q = 3$, i.e. for the modular group $\text{PSL}_2(\mathbb{Z})$, and χ being the trivial representation, Theorem 3.4 was already proven in [12] and [6]. In this case, the set D_0 is empty and hence there is no f_0 -component, and the isomorphism map simplifies to

$$f_{-1} = \alpha_s(g_1)\varphi, \quad \varphi = \alpha_s(g_1^{-1})f_{-1}.$$

For the proof of Theorem 3.4 we first show an immediate result of own interest on the maximal domain of holomorphy for the elements of $\text{SEF}_{q,s}^{\omega,\pm}$ and $\text{FEF}_{q,s}^{\pm}$. To simplify notation, we omit the subscript q .

Let

$$A := \{g_{\pm 1}^{-1}, \dots, g_{\pm m}^{-1}\}$$

be the elements acting in the transfer operators (the ‘alphabet’). For each $n \in \mathbb{N}_0$, let

$$A^n := \left\{ g_{k_1}^{-1} \cdots g_{k_n}^{-1} \mid g_{k_j}^{-1} \in A \text{ for } j = 1, \dots, n \right\}$$

denote the ‘words’ of length n over A , and let

$$A^* := \bigcup_{n \in \mathbb{N}_0} A^n$$

denote the set of all words over A . Further let

$$\begin{aligned} A_{-1}^n &:= \{g_{k_1}^{-1} \cdots g_{k_n}^{-1} \in A^n \mid k_1 = -1\}, \\ A_{(-1,1)}^n &:= \{g_{k_1}^{-1} \cdots g_{k_n}^{-1} \in A^n \mid k_n = 1\}, \\ A_{(-1,-1)}^n &:= \{g_{k_1}^{-1} \cdots g_{k_n}^{-1} \in A^n \mid k_n = -1\}, \end{aligned}$$

and

$$\begin{aligned} A_0^n &:= \{g_{k_1}^{-1} \cdots g_{k_n}^{-1} \in A^n \mid k_1 \in \{-2, \dots, -m\}\}, \\ A_{(0,1)}^n &:= \{g_{k_1}^{-1} \cdots g_{k_n}^{-1} \in A^n \mid k_n = 1\}, \\ A_{(0,-1)}^n &:= \{g_{k_1}^{-1} \cdots g_{k_n}^{-1} \in A^n \mid k_n = -1\}, \end{aligned}$$

as well as

$$A_{-1}^* := \bigcup_{n \in \mathbb{N}_0} A_{-1}^n, \quad A_{(-1,1)}^* := \bigcup_{n \in \mathbb{N}_0} A_{(-1,1)}^n, \quad A_{(-1,-1)}^* := \bigcup_{n \in \mathbb{N}_0} A_{(-1,-1)}^n$$

and

$$A_0^* := \bigcup_{n \in \mathbb{N}_0} A_0^n, \quad A_{(0,1)}^* := \bigcup_{n \in \mathbb{N}_0} A_{(0,1)}^n, \quad A_{(0,-1)}^* := \bigcup_{n \in \mathbb{N}_0} A_{(0,-1)}^n$$

Let

$$\mathbb{C}_R := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}.$$

Lemma 3.6. *Let \mathcal{U}_{-1} be a complex neighborhood of D_{-1} , and \mathcal{U}_0 a complex neighborhood of D_0 . Let $\mathcal{U} \subseteq \mathbb{C}$ be an open bounded set that is bounded away from $(-\infty, 0]$, and let $\mathcal{V} \subseteq \mathbb{C}$ be an open bounded set that is bounded away from $(-\infty, -1/\ell]$. Then the following properties are satisfied.*

- (i) *There are only finitely many $g \in A_{-1}^*$ for which $g\mathcal{U} \not\subseteq \mathcal{U}_{-1}$ or $gQ\mathcal{U} \not\subseteq \mathcal{U}_{-1}$. Moreover, $g(\mathcal{U}_{-1} \cap \mathbb{C}_R) \subseteq \mathcal{U}_{-1} \cap \mathbb{C}_R$.*
- (ii) *There are only finitely many $g \in A_0^*$ for which $g\mathcal{U} \not\subseteq \mathcal{U}_0$ or $gQ\mathcal{U} \not\subseteq \mathcal{U}_0$. Moreover, $g(\mathcal{U}_0 \cap \mathbb{C}_R) \subseteq \mathcal{U}_0 \cap \mathbb{C}_R$.*
- (iii) *There are only finitely many $g \in A_0^* \setminus A_{(0,-1)}^*$ for which $g\mathcal{V} \not\subseteq \mathcal{U}_0$. Moreover, $g(\mathcal{U}_0 \cap \mathbb{C}_R) \subseteq \mathcal{U}_0 \cap \mathbb{C}_R$.*
- (iv) *There are only finitely many $g \in A_0^* \setminus A_{(0,1)}^*$ for which $gQ\mathcal{V} \not\subseteq \mathcal{U}_0$. Moreover, $g(\mathcal{U}_0 \cap \mathbb{C}_R) \subseteq \mathcal{U}_0 \cap \mathbb{C}_R$.*
- (v) *There are only finitely many $g \in A_{-1}^* \setminus A_{(-1,-1)}^*$ for which $g\mathcal{V} \not\subseteq \mathcal{U}_{-1}$. Moreover, $g(\mathcal{U}_{-1} \cap \mathbb{C}_R) \subseteq \mathcal{U}_{-1} \cap \mathbb{C}_R$.*
- (vi) *There are only finitely many $g \in A_{-1}^* \setminus A_{(-1,1)}^*$ for which $gQ\mathcal{V} \not\subseteq \mathcal{U}_{-1}$. Moreover, $g(\mathcal{U}_{-1} \cap \mathbb{C}_R) \subseteq \mathcal{U}_{-1} \cap \mathbb{C}_R$.*

Proof. We only show (i) as the proofs of the remaining statements are analogous. The proof of (i) can be read off from Figures 3 and 4. For a more detailed proof we refer to [19, 22]. Figure 3 indicates the location of $g\mathbb{C}_R$ for $g \in A^*$. It shows that if \mathcal{U} is contained in \mathbb{C}_R then $h\mathcal{U} \subseteq \mathcal{U}_{-1}$ for all sufficiently long words $h \in A_{-1}^*$. Since \mathbb{C}_R is invariant under the action of Q , and \mathcal{U} is bounded away from $Q \cdot [-\infty, 0] = [-\infty, 0]$, it also follows $hQ\mathcal{U} \subseteq \mathcal{U}_{-1}$ for all sufficiently long words $h \in A_{-1}^*$. Figure 4 indicates the location of $g^{-1}\mathbb{C}_R$ for $g \in A^*$. We remark that

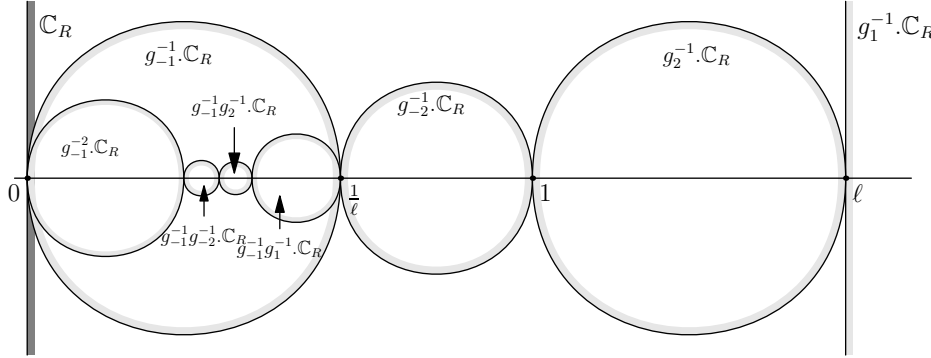
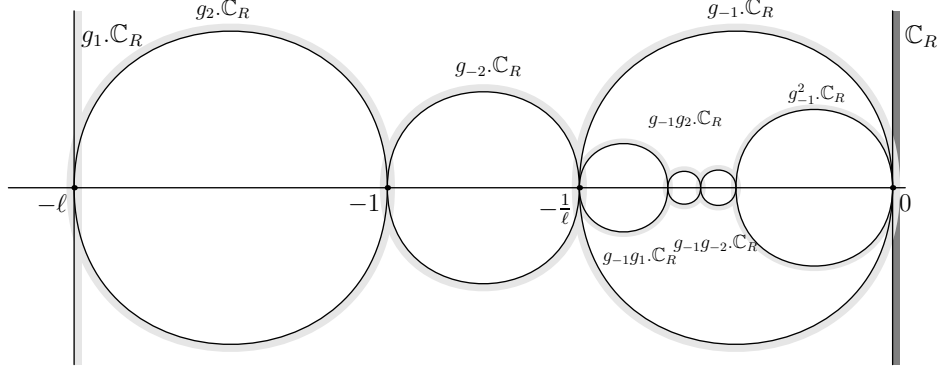


FIGURE 3. Images of \mathbb{C}_R under A^* for $q = 5$.

for each $n \in \mathbb{N}$, the set

$$V_n := \bigcap_{g \in A^n} g^{-1}\mathbb{C}_R$$

FIGURE 4. Images of \mathbb{C}_R under $(A^*)^{-1}$ for $q = 5$.

is nonempty, and even more,

$$V_n \subseteq V_{n+1}$$

as well as

$$\{z \in \mathbb{C} \mid \operatorname{Re} z < 0, \operatorname{Im} z \neq 0\} \subseteq \bigcup_{n \in \mathbb{N}} V_n.$$

Hence, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $g \in A^n$, $g\mathcal{U} \subseteq \mathbb{C}_R$. This completes the proof. \square

For $n \in \mathbb{N}_0$ let

$$A_L^n := A_{-1}^n \cup A_0^n.$$

Set

$$\mathbb{C}_R^* := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\} \quad \text{and} \quad \mathbb{C}' := \mathbb{C} \setminus (-\infty, 0].$$

Proposition 3.7. *Let $s \in \mathbb{C}$ and $\varphi \in \operatorname{SEF}_s^{\omega, \pm}$. If we use $\alpha_s^{(1)}$ for α_s then φ extends holomorphically to \mathbb{C}_R^* and satisfies*

$$(12) \quad \varphi = \sum_{k=1}^m (\alpha_s(g_{-k}) \pm \alpha_s(Qg_{-k}))\varphi$$

on all of \mathbb{C}_R^ . If we use $\alpha_s^{(2)}$ for α_s then φ extends holomorphically to \mathbb{C}' and satisfies (12) on \mathbb{C}' .*

Proof. By hypothesis, $\varphi: (0, 1) \rightarrow \mathbb{C}$ is real-analytic. Thus, there exists a complex neighborhood \mathcal{U} of $(0, 1)$ such that φ extends holomorphically to \mathcal{U} . Without loss of generality, we may assume that for $k = 1, \dots, m$, $g_{-k}^{-1}\mathcal{U} \subseteq \mathcal{U}$ and $g_{-k}^{-1}Q\mathcal{U} \subseteq \mathcal{U}$. Thus, the identity theorem of complex analysis implies that the functional equation

$$\varphi = \mathcal{L}_s^{\text{slow}, \pm} \varphi = \sum_{k=1}^m (\alpha_s(g_{-k}) \pm \alpha_s(Qg_{-k}))\varphi$$

remains valid on all of \mathcal{U} . Even more, for any $n \in \mathbb{N}$ we have

$$(13) \quad \varphi = (\mathcal{L}_s^{\text{slow}, \pm})^n \varphi = \left(\sum_{a \in A_L^n} \alpha_s(a^{-1}) \pm \alpha_s(Qa^{-1}) \right) \varphi$$

on $(0, 1)$, and hence (cf. Lemma 3.6) on \mathcal{U} .

For $\alpha_s^{(1)}$ note that \mathbb{C}_R^* is the largest domain that contains $(0, 1)$ and on which all the cocycles in (13) are well-defined. Let $z_0 \in \mathbb{C}_R^*$ and fix an open bounded neighborhood \mathcal{W} of z_0 in \mathbb{C}_R^* . By Lemma 3.6 there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ and $g \in A_L^n$ we have $g\mathcal{W} \subseteq \mathcal{U}$ and $gQ\mathcal{W} \subseteq \mathcal{U}$. We fix $n \geq n_0$ and define

$$(14) \quad \varphi := \left(\sum_{a \in A_L^n} \alpha_s(a^{-1}) \pm \alpha_s(Qa^{-1}) \right) \varphi \quad \text{on } \mathcal{W}.$$

In order to see that the left hand side of (14) is well-defined let $m \geq n_0$. Without loss of generality, we may suppose that $m > n$. Obviously, on \mathcal{U} we have

$$\varphi = (\mathcal{L}_s^{\text{slow}, \pm})^{m-n} \varphi.$$

Thus, by using (13) and (14) we find on all of \mathcal{W} the identity

$$\begin{aligned} & \left(\sum_{a \in A_L^n} \alpha_s(a^{-1}) \pm \alpha_s(Qa^{-1}) \right) \varphi \\ &= \left(\sum_{a \in A_L^n} \alpha_s(a^{-1}) \pm \alpha_s(Qa^{-1}) \right) \left(\sum_{b \in A_L^{m-n}} \alpha_s(b^{-1}) \pm \alpha_s(Qb^{-1}) \right) \varphi \\ &= \left(\sum_{a \in A_L^n} \sum_{b \in A_L^{m-n}} \alpha_s(a^{-1}b^{-1}) \pm \alpha_s(Qa^{-1}b^{-1}) \pm \alpha_s(a^{-1}Qb^{-1}) + \alpha_s(Qa^{-1}Qb^{-1}) \right) \varphi \\ &= \left(\sum_{c \in A_L^m} \alpha_s(c^{-1}) \pm \alpha_s(Qc^{-1}) \right) \varphi. \end{aligned}$$

This shows well-definedness. Clearly, each summand of the right hand side of (14) is holomorphic on \mathcal{W} , hence φ is holomorphic on \mathcal{W} as well. Finally, φ satisfies (12) on \mathcal{W} since

$$\begin{aligned} & \left(\sum_{k=1}^m \alpha_s(g_{-k}) \pm \alpha_s(Qg_{-k}) \right) \varphi \\ &= \left(\sum_{k=1}^m \alpha_s(g_{-k}) \pm \alpha_s(Qg_{-k}) \right) \left(\sum_{a \in A_L^n} \alpha_s(a^{-1}) \pm \alpha_s(Qa^{-1}) \right) \varphi \\ &= \left(\sum_{k=1}^m \sum_{a \in A_L^n} \alpha_s(g_{-k}a^{-1}) \pm \alpha_s(g_{-k}Qa^{-1}) \pm \alpha_s(Qg_{-k}a^{-1}) + \alpha_s(Qg_{-k}Qa^{-1}) \right) \varphi \\ &= \left(\sum_{k=1}^m \sum_{a \in A_L^n} \alpha_s(g_{-k}a^{-1}) + \alpha_s(g_k a^{-1}) \pm \alpha_s(Qg_k a^{-1}) \pm \alpha_s(Qg_{-k} a^{-1}) \right) \varphi \\ &= \left(\sum_{b \in A_L^{n+1}} \alpha_s(b^{-1}) \pm \alpha_s(Qb^{-1}) \right) \varphi \end{aligned}$$

$$= \varphi.$$

This completes the proof for $\alpha_s^{(1)}$. The proof for $\alpha_s^{(2)}$ is analogous. \square

Let

$$B := \{g_{\pm 1}^{-p}, g_{\pm 2}^{-1}, \dots, g_{\pm m}^{-1} \mid p \in \mathbb{N}\}.$$

We call a word over the alphabet B *reduced* if it does not contain a subword of the form $g_1^{-p_1} g_1^{-p_2}$ or $g_{-1}^{-p_1} g_{-1}^{-p_2}$ with $p_1, p_2 \in \mathbb{N}$. For each $n \in \mathbb{N}_0$, let

$$B^n := \{h_{k_1} \cdots h_{k_n} \mid h_{k_j} \in B \text{ for } j = 1, \dots, n\}$$

denote the set of reduced words of length n over B . Further let

$$\begin{aligned} B_0^n &:= \{h_{k_1} \cdots h_{k_n} \in B^n \mid k_1 \in \{-2, \dots, -m\}\}, \\ B_{(0,1)}^n &:= \{h_{k_1} \cdots h_{k_n} \in B_0^n \mid k_n = 1\}, \\ B_{-1}^n &:= \{h_{k_1} \cdots h_{k_n} \in B^n \mid k_1 = -1\}, \\ B_{(-1,-1)}^n &:= \{h_{k_1} \cdots h_{k_n} \in B_{-1}^n \mid k_n = -1\} \end{aligned}$$

and

$$B_{(-1,1)}^n := \{h_{k_1} \cdots h_{k_n} \in B_{-1}^n \mid k_n = 1\}.$$

Proposition 3.8. *Let $s \in \mathbb{C}$ and $f = (f_0, f_{-1})^\top \in \text{FEF}_s^\pm$. If we use $\alpha_s^{(1)}$ for α_s then f_0 extends holomorphically to \mathbb{C}_R^* and f_{-1} extends holomorphically to $\mathbb{C}_\ell^* := \{z \in \mathbb{C} \mid \text{Re } z > -1/\ell\}$. The holomorphically extended function vector $f = (f_0, f_{-1})^\top$ satisfies*

$$(15) \quad f = \begin{pmatrix} (1 \pm \alpha_s(Q)) \mathcal{L}_{0,s}^{\text{fast}} & (1 \pm \alpha_s(Q)) \mathcal{L}_{-1,s}^{\text{fast}} \\ (1 \pm \alpha_s(Q)) \mathcal{L}_{0,s}^{\text{fast}} & \pm \alpha_s(Q) \mathcal{L}_{-1,s}^{\text{fast}} \end{pmatrix} f.$$

If we use $\alpha_s^{(2)}$ for α_s then f_0 extends holomorphically to \mathbb{C}' and f_{-1} extends holomorphically to $\mathbb{C} \setminus (-\infty, -1/\ell]$, and the function vector $(f_0, f_{-1})^\top$ satisfies (15).

Proof. It suffices to show the proposition for $\text{Re } s > 1/2$. We only provide the proof for $\alpha_s^{(1)}$ as the consideration of $\alpha_s^{(2)}$ is analogous. We note that $\mathbb{C}_R^* \times \mathbb{C}_\ell^*$ is the maximal domain of holomorphy that contains $D_0 \times D_{-1}$ and on which all arising cocycles are simultaneously well-defined.

For $n \in \mathbb{N}_0$ we have (cf. [13])

$$(\mathcal{L}_s^{\text{fast}, \pm})^n = \begin{pmatrix} (1 \pm \alpha_s(Q)) \sum_{b \in B_0^n} \alpha_s(b^{-1}) & (1 \pm \alpha_s(Q)) \sum_{b \in B_{-1}^n} \alpha_s(b^{-1}) \\ \sum_{b \in B_0^n \setminus B_{(0,-1)}^n} \alpha_s(b^{-1}) \pm \sum_{b \in B_0^n \setminus B_{(0,1)}^n} \alpha_s(Qb^{-1}) & \sum_{b \in B_{-1}^n \setminus B_{(-1,-1)}^n} \alpha_s(b^{-1}) \pm \sum_{b \in B_{-1}^n \setminus B_{(-1,1)}^n} \alpha_s(Qb^{-1}) \end{pmatrix}.$$

Let $(z_0, w_0) \in \mathbb{C}_R^* \times \mathbb{C}_\ell^*$ and pick open bounded neighborhoods \mathcal{U} of z_0 in \mathbb{C}_R^* and \mathcal{V} of w_0 in \mathbb{C}_ℓ^* . Further, for $j \in \{-1, 0\}$, let \mathcal{D}_j be open complex neighborhoods of $\overline{D_j}$ such that $f \in B(\mathcal{D}_0) \oplus B(\mathcal{D}_{-1})$.

By Lemma 3.6 there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$g\mathcal{U} \subseteq \mathcal{D}_0 \quad \text{and} \quad gQ\mathcal{U} \subseteq \mathcal{D}_0$$

for all $g \in B_0^n$, and

$$g \cdot \mathcal{V} \subseteq \mathcal{D}_{-1} \quad \text{and} \quad gQ \cdot \mathcal{V} \subseteq \mathcal{D}_{-1}$$

for all $g \in B_{-1}^n$. We fix $n \geq n_0$ and define

$$(16) \quad \begin{pmatrix} f_0 \\ f_{-1} \end{pmatrix} := (\mathcal{L}_s^{\text{fast}, \pm})^n \begin{pmatrix} f_0 \\ f_{-1} \end{pmatrix}$$

on $\mathcal{U} \times \mathcal{V}$. As in the proof of Proposition 3.7 we see that the left hand side of (14) is well-defined and defines a holomorphic function vector that satisfies (15) on $\mathcal{U} \times \mathcal{V}$. \square

As a second intermediate result we show that

$$\mathcal{L}_{-1,s}^{\text{fast}} f_{-1} = \alpha_s(g_{-1})\varphi$$

whenever $f = (f_0, f_{-1})^\top \in \text{FEF}_s^\pm$ is given and φ is defined by (10), or $\varphi \in \text{SEF}_s^{\omega, \text{as}, \pm}$ is given and f is defined by (11).

To that end let

$$E_1 := \{v \in V \mid \chi(g_{-1})v = v\},$$

let E_r be the orthogonal complement of E_1 in V , and define

$$\text{pr}_r : V \rightarrow E_r$$

to be the orthogonal projection on E_r . Further, we consider the Lerch zeta function $\zeta(s, a, x)$ for $x > 0$ only (in which case $\alpha_s = \alpha_s^{(1)} = \alpha_s^{(2)}$) and recall that its asymptotic expansion for $x \rightarrow \infty$ is

$$(17) \quad \zeta(s, a, x) \sim \sum_{n=-1}^{\infty} D_n x^{-(s+n)}$$

for certain coefficients $D_n \in \mathbb{C}$, $n \in \mathbb{Z}_{>-1}$, depending on s and a with $D_{-1} = 0$ if $a \notin \mathbb{Z}$ [11]. The precise values for all D_n are known [11] but are not of importance for us.

Proposition 3.9. *Let $s \in \mathbb{C}$ and $f = (f_0, f_{-1})^\top \in \text{FEF}_s^\pm$. Then*

- (i) $\alpha_s(g_{-1}) \circ (1 + \mathcal{L}_{-1,s}^{\text{fast}}) f_{-1} = \mathcal{L}_{-1,s}^{\text{fast}} f_{-1}$ on $\mathbb{R}_{>0}$.
- (ii) $(1 + \mathcal{L}_{-1,s}^{\text{fast}}) f_{-1}(x) = \frac{c}{x} + O_{x \rightarrow 0^+}(1)$ for some $c = c(s, f) \in V$. Moreover, $\text{pr}_r(c) = 0$.

Proof. To simplify notation, we set $\mathcal{L}_s := \mathcal{L}_{-1,s}^{\text{fast}}$. We start with a diagonalization. Since $\chi(g_{-1}^{-1})$ is a unitary operator on V , there exists an orthonormal basis of V with respect to which $\chi(g_{-1}^{-1})$ is represented by a unitary diagonal matrix, say

$$\text{diag}(e^{2\pi i a_1}, \dots, e^{2\pi i a_d})$$

with $a_1, \dots, a_d \in \mathbb{R}$ and $d = \dim V$. We use the same basis of V to represent any function $\psi : D \rightarrow V$ (here, D is any domain that arise in our considerations) as a vector of component functions

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix} : D \rightarrow \mathbb{C}^d.$$

For $s \in \mathbb{C}$, $g \in G$, any subset I of \mathbb{R} and any function $f: I \rightarrow \mathbb{C}$ we set

$$\tau_s(g^{-1})f(x) := |g'(x)|^s f(g.x),$$

whenever it makes sense. Then, in these coordinates for V and for $\operatorname{Re} s > \frac{1}{2}$, the operator \mathcal{L}_s acts as

$$\operatorname{diag} \left(\sum_{n \in \mathbb{N}} e^{2\pi i n a_1} \tau_s(g_{-1}^n), \dots, \sum_{n \in \mathbb{N}} e^{2\pi i n a_d} \tau_s(g_{-1}^n) \right).$$

We now consider a single component. Let $a \in \mathbb{R}$ and, by a slight abuse of notation,

$$\alpha_s(g_{-1}) := \alpha_s^{\mathbb{C}}(g_{-1}) := e^{2\pi i a} \tau_s(g_{-1}).$$

For $\operatorname{Re} s > \frac{1}{2}$ let

$$(18) \quad L_s := \sum_{n \in \mathbb{N}} \alpha_s(g_{-1}^n) = \sum_{n \in \mathbb{N}} e^{2\pi i n a} \tau_s(g_{-1}^n),$$

and let h be a smooth complex-valued function that is defined in some neighborhood of 0. For $k \in \mathbb{N}_0$ let

$$c_k := \frac{h^{(k)}(0)}{k!} \quad \text{and} \quad h_k(x) := c_k x^k.$$

Let $M \in \mathbb{N}_0$. In order to state L_s 's meromorphic continuation to $\operatorname{Re} s > (1 - M)/2$ we define

$$P_M(h)(x) := h(x) - \sum_{k=0}^{M-1} c_k x^k$$

and $Q_M := 1 - P_M$. Then

$$L_s = L_s \circ Q_M + L_s \circ P_M,$$

where $L_s \circ P_M$ converges for $\operatorname{Re} s > (1 - M)/2$ and the meromorphic continuation of $L_s \circ Q_M$ is given by

$$(L_s \circ Q_M)h: x \mapsto \frac{e^{2\pi i a}}{(\ell x)^{2s}} \sum_{k=0}^{M-1} c_k \zeta \left(2s + k, a, 1 + \frac{1}{\ell x} \right).$$

For the proof of (i) note that

$$(\alpha_s(g_{-1}) \circ L_s \circ Q_M)h(x) = \frac{e^{2\pi i 2a}}{(\ell x)^{2s}} \sum_{k=0}^{M-1} c_k \zeta \left(2s + k, a, 2 + \frac{1}{\ell x} \right)$$

and

$$(\alpha_s(g_{-1}) \circ L_s \circ P_M)h = L_s \circ P_M h + L_s \circ Q_M h - \alpha_s(g_{-1})P_M h - L_s \circ Q_M h.$$

Thus,

$$\begin{aligned} \alpha_s(g_{-1})L_s h(x) &= \alpha_s(g_{-1})L_s P_M h(x) + \alpha_s(g_{-1})L_s Q_M h(x) \\ &= L_s h(x) - \alpha_s(g_{-1})h(x) + \sum_{k=0}^{M-1} \frac{c_k e^{2\pi i a}}{(\ell x)^{2s}} \left[\left(1 + \frac{1}{\ell x} \right)^{-(2s+k)} \right. \\ &\quad \left. - \zeta \left(2s + k, a, 1 + \frac{1}{\ell x} \right) + e^{2\pi i a} \zeta \left(2s + k, a, 2 + \frac{1}{\ell x} \right) \right] \end{aligned}$$

$$= L_s h(x) - \alpha_s(g_{-1})h(x).$$

This proves (i).

For (ii) we note that

$$(1 + L_s)h(x) \sim \frac{1}{(\ell x)^{2s}} \sum_{k=0}^{\infty} c_k \zeta\left(2s + k, a, \frac{1}{\ell x}\right) \quad \text{as } x \rightarrow 0^+.$$

Combining this with the asymptotic expansion (17) yields

$$(1 + L_s)h(x) \sim \frac{1}{(\ell x)^{2s}} \sum_{k=0}^{\infty} c_k \sum_{n=-1}^{\infty} d_n(k) (\ell x)^{2s+k+n} = \sum_{p=-1}^{\infty} c_p^* x^p$$

as $x \rightarrow 0^+$ for appropriate coefficients $d_n(k) \in \mathbb{C}$ (depending on s, a, k) and c_p^* (depending on the c_k 's and $d_n(k)$'s). Moreover, $c_{-1}^* = 0$ if $a \notin \mathbb{Z}$. This completes the proof. \square

Proposition 3.10. *Let $s \in \mathbb{C}$ and $\varphi \in \text{SEF}_s^{\omega, \pm}$. Set*

$$(19) \quad \psi := (1 - \alpha_s(g_{-1}))\varphi = \mathcal{L}_s^{\text{slow}, \pm} \varphi - \alpha_s(g_{-1})\varphi.$$

Then

$$Q_0 := \alpha_s(g_{-1})\varphi - \mathcal{L}_{-1,s}^{\text{fast}} \psi : (0, 1) \rightarrow V$$

is a real-analytic $\alpha_s(g_{-1})$ -invariant function. Further, φ has an asymptotic expansion of the form

$$\varphi(x) \sim Q_0(x) + \sum_{n=-1}^{\infty} C_n^* x^n \quad \text{as } x \rightarrow 0^+$$

for certain (unique) coefficients $C_n^ \in V$, $n \in \mathbb{Z}_{\geq -1}$. Moreover, $\text{pr}_r(C_{-1}^*) = 0$.*

Proof. Obviously, Q_0 is real-analytic. We start by showing that Q_0 is $\alpha_s(g_{-1})$ -invariant. To that end let f be an arbitrary function which is smooth in a neighborhood of 0. To simplify notation, we set

$$\mathcal{L}_s := \mathcal{L}_{-1,s}^{\text{fast}}.$$

For $\text{Re } s > \frac{1}{2}$ we have

$$(20) \quad \alpha_s(g_{-1})\mathcal{L}_s f = \mathcal{L}_s f - \alpha_s(g_{-1})f.$$

Since f is arbitrary (hence, in particular, independent of s), meromorphic continuation in s shows that (20) holds for all $s \in \mathbb{C} \setminus \{\text{poles}\}$. Thus, applying (20) with $f = \psi$ and recalling (19) yields

$$\begin{aligned} \alpha_s(g_{-1})Q_0 &= \alpha_s(g_{-1}^2)\varphi - \alpha_s(g_{-1})\mathcal{L}_s \psi \\ &= \alpha_s(g_{-1}^2)\varphi - \mathcal{L}_s \psi + \alpha_s(g_{-1})\psi \\ &= \alpha_s(g_{-1}^2)\varphi - \mathcal{L}_s \psi + \alpha_s(g_{-1})\varphi - \alpha_s(g_{-1}^2)\varphi \\ &= -\mathcal{L}_s \psi + \alpha_s(g_{-1})\varphi \\ &= Q_0. \end{aligned}$$

Hence, Q_0 is $\alpha_s(g_{-1})$ -invariant.

For the asymptotic expansion we note that

$$(21) \quad \varphi = Q_0 + \psi + \mathcal{L}_s \psi.$$

As in the proof of Proposition 3.9 we find that the asymptotic expansion of $\psi + \mathcal{L}_s \psi$ for $x \rightarrow 0^+$ is of the claimed form. \square

Lemma 3.11. *Let $s \in \mathbb{C}$ and let Q_0 be as in Proposition 3.10. Then we have*

- (i) For $\operatorname{Re} s > \frac{1}{2}$ and $\varphi = o(x^{-2s})$, $Q_0 = 0$.
- (ii) $Q_0(x) = O_{x \rightarrow 0^+}(x^{-2s})$.
- (iii) If $Q_0(x) = o_{x \rightarrow 0^+}(x^{-2s})$ then $Q_0 = 0$.
- (iv) Let $\frac{1}{2} \geq \operatorname{Re} s > 0$, $s \neq \frac{1}{2}$. If

$$(22) \quad Q_0(x) = \frac{c}{x} + O(1) \quad \text{as } x \rightarrow 0^+$$

then $c = 0$.

Proof. For (i) recall that, for $\operatorname{Re} s > \frac{1}{2}$, the operator $\mathcal{L}_{-1,s}^{\text{fast}}$ is given by (8). A straightforward calculation shows $Q_0 = 0$.

The $\alpha_s(g_{-1})$ -invariance of Q_0 easily implies (iii). For (ii) and (iv) note that the map

$$\tilde{Q}_0 := \alpha_s(Q)Q_0 : (1, \infty) \rightarrow \mathbb{C}$$

is a real-analytic $\alpha_s(g_1)$ -invariant function. In particular, \tilde{Q}_0 is bounded. Thus,

$$Q_0(x) = \alpha_s(Q)\tilde{Q}_0(x) = x^{-2s}\tilde{Q}_0\left(\frac{1}{x}\right) \ll |x^{2s}|.$$

This proves (ii). For (iv) note that (22) is equivalent to

$$(23) \quad \tilde{Q}_0(x) = cx^{1-2s} + O(x^{-2s}) \quad \text{as } x \rightarrow \infty.$$

Thus, for $\frac{1}{2} > \operatorname{Re} s > 0$ it follows that \tilde{Q}_0 is unbounded unless $c = 0$. Hence the boundedness of \tilde{Q}_0 implies $c = 0$. It remains to consider the case $\operatorname{Re} s = \frac{1}{2}$. Let

$$t := -2 \operatorname{Im} s$$

and note that $t \neq 0$. The $\alpha_s(g_1)$ -invariance of \tilde{Q}_0 shows that for each $x \in (1, \infty)$ and $k \in \mathbb{N}$ we have

$$|c| |x^{it} - (x + k\ell)^{it}| \leq \left| \tilde{Q}_0(x) - cx^{it} \right| + \left| \tilde{Q}_0(x + k\ell) - c(x + k\ell)^{it} \right|.$$

Thus, the growth condition (23) yields that

$$(24) \quad |c| |x^{it} - (x + k\ell)^{it}| \rightarrow 0 \quad \text{as } x \rightarrow \infty, k \rightarrow \infty.$$

We have

$$|x^{it} - (x + k\ell)^{it}| = \left| \exp\left(-it \log\left(1 + \frac{k\ell}{x}\right)\right) - 1 \right|.$$

For all $k_0 \in \mathbb{N}$, $x_0 > 1$,

$$\left\{ \frac{k}{x} \mid k \geq k_0, x \geq x_0 \right\} = (0, \infty).$$

Hence,

$$\limsup_{x \rightarrow \infty, k \rightarrow \infty} \left| \exp \left(-it \log \left(1 + \frac{k}{x} \right) \right) - 1 \right| = 2.$$

In turn, the convergence (24) is only possible for $c = 0$. This completes the proof. \square

Corollary 3.12. *Let $s \in \mathbb{C}$, $\operatorname{Re} s > 0$, $s \neq 1/2$. Suppose that $\varphi \in \operatorname{SEF}_s^{\omega, \text{as}, \pm}$ and define ψ as in (19). Then*

$$\alpha_s(g_{-1})\varphi = \mathcal{L}_{-1,s}^{\text{fast}}\psi.$$

Proof. The combination of Lemma 3.11 with the asymptotic expansion for φ from Proposition 3.10 and the supposed growth of φ immediately yields a proof. \square

Proof of Theorem 3.4. Suppose first that $\varphi \in \operatorname{SEF}_s^{\omega, \text{as}, \pm}$ and define $f = (f_0, f_{-1})^\top$ as in (11). By Proposition 3.7, φ extends holomorphically to \mathbb{C}_R^* and satisfies (12) on \mathbb{C}_R^* . Thus, the definition of f_0 extends holomorphically to \mathbb{C}_R^* . Further, taking advantage of (12), we find that

$$f_{-1} = (1 - \alpha_s(g_{-1}))\varphi = \sum_{k=2}^m (\alpha_s(g_{-k}) \pm \alpha_s(Qg_{-k}))\varphi \pm \alpha_s(Qg_{-1})\varphi$$

is in fact defined and holomorphic on \mathbb{C}_ℓ^* . By the identity theorem of complex analysis, it suffices to show that f satisfies $f = \mathcal{L}_s^{\text{fast}, \pm}$ on $D_0 \times D_{-1}$. Corollary 3.12 shows $\mathcal{L}_{-1,s}^{\text{fast}}f_{-1} = \alpha_s(g_{-1})\varphi$ on $\mathbb{R}_{>0}$.

In particular,

$$(1 \pm \alpha_s(Q))\mathcal{L}_{-1,s}^{\text{fast}}f_{-1} = (\alpha_s(g_{-1}) \pm \alpha_s(Qg_{-1}))\varphi.$$

Analogously, on all of $\mathbb{R}_{>0}$ we have

$$\begin{aligned} (1 \pm \alpha_s(Q))\mathcal{L}_{0,s}^{\text{fast}}f_0 &= (1 \pm \alpha_s(Q))\mathcal{L}_{0,s}^{\text{fast}}\varphi \\ &= \mathcal{L}_s^{\text{slow}, \pm}\varphi - (\alpha_s(g_{-1}) \pm \alpha_s(Qg_{-1}))\varphi. \end{aligned}$$

Then a straightforward calculation shows

$$\mathcal{L}_s^{\text{fast}, \pm}f = f.$$

If φ satisfies (6) then f obviously satisfies (9).

Suppose now that $f = \mathcal{L}_s^{\text{fast}, \pm}f$ and define φ as in (10). Since f_0 and f_{-1} are holomorphic in a complex neighborhood of $\overline{D_0}$ respectively of $\overline{D_{-1}}$, φ is real-analytic on $(0, 1)$ and even holomorphic in a complex neighborhood of $(0, 1)$. Therefore it suffices to show that φ satisfies $\varphi = \mathcal{L}_s^{\text{slow}, \pm}\varphi$ on $D_{-1} \cup D_0$. By Proposition 3.9(i) we have $\alpha_s(g_{-1})\varphi = \mathcal{L}_{-1,s}^{\text{fast}}f_{-1}$ on $\mathbb{R}_{>0}$. Then $f = \mathcal{L}_s^{\text{fast}, \pm}f$ yields that on D_0 ,

$$\begin{aligned} \varphi|_{D_0} = f_0 &= (1 \pm \alpha_s(Q))\mathcal{L}_{0,s}^{\text{fast}}f_0 + (1 \pm \alpha_s(Q))\mathcal{L}_{-1,s}^{\text{fast}}f_{-1} \\ &= (1 \pm \alpha_s(Q)) \sum_{k=2}^m \alpha_s(g_{-k})\varphi + (1 \pm \alpha_s(Q))\alpha_s(g_{-1})\varphi \\ &= \mathcal{L}_s^{\text{slow}, \pm}\varphi. \end{aligned}$$

On D_{-1} we have

$$\varphi|_{D_{-1}} = f_{-1} + \mathcal{L}_{-1,s}^{\text{fast}}f_{-1}$$

$$\begin{aligned}
&= (1 \pm \alpha_s(Q)) \mathcal{L}_{0,s}^{\text{fast}} f_0 \pm \alpha_s(Q) \mathcal{L}_{-1,s}^{\text{fast}} f_{-1} + \mathcal{L}_{-1,s}^{\text{fast}} f_{-1} \\
&= \mathcal{L}_s^{\text{slow}, \pm} \varphi.
\end{aligned}$$

This shows $\mathcal{L}_s^{\text{slow}, \pm} \varphi$ for φ . Then Proposition 3.9(ii) yields $\varphi \in \text{SEF}_s^{\omega, \text{as}, \pm}$. Finally, if f satisfies (9) then φ clearly satisfies (6). \square

3.1.2. *The case of even q .* For even q the statements and proofs are almost identical to those for odd q . The necessary changes are caused by the fact that

$$g_{q, \frac{q}{2}} = g_{q, -\frac{q}{2}},$$

and the attracting fixed point of $g_{q, q/2}^{-1}$ is 1.

The odd resp. even slow transfer operator $\mathcal{L}_{q,s}^{\text{slow}, \pm}$ of Γ_q is given by

$$\begin{aligned}
\mathcal{L}_{q,s}^{\text{slow}, \pm} &= \frac{1}{2} \alpha_s(g_{q, q/2}) \pm \frac{1}{2} \alpha_s(Qg_{q, q/2}) + \sum_{k=1}^m \alpha_s(g_{q, -k}) \pm \alpha_s(Qg_{q, -k}) \\
&= (1 \pm \alpha_s(Q)) \left(\frac{1}{2} \alpha_s(g_{q, q/2}) + \sum_{k=1}^m \alpha_s(g_{q, -k}) \right).
\end{aligned}$$

We consider it to act on $C^\omega((0, 1+\varepsilon); V)$ for some $\varepsilon > 0$ (or equivalently $C^\omega(\mathbb{R}_{>0}; V)$). Likewise, the spaces $\text{SEF}_{q,s}^{\omega, \pm}$, $\text{SEF}_{q,s}^{\omega, \text{as}, \pm}$ and $\text{SEF}_{q,s}^{\omega, \text{dec}, \pm}$ are defined for functions in $C^\omega((0, 1+\varepsilon); V)$.

For the odd resp. even fast transfer operators we need to use

$$\mathcal{L}_{q,0,s}^{\text{fast}} := \frac{1}{2} \alpha_s(g_{q, q/2}) + \sum_{k=2}^m \alpha_s(g_{q, -k})$$

and set

$$D_0 := \left(\frac{1}{\ell(q)}, 1 \right].$$

With these changes the statement and proof of Theorem 3.4 applies for even q as well.

3.2. **Isomorphism for the Theta group.** For the Theta group

$$\Gamma := \Gamma_2$$

we consider the slow and fast transfer operators that are developed in [15]. Let

$$k_1 := \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad k_2 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

The even resp. odd slow transfer operator for Γ is

$$\mathcal{L}_s^{\text{slow}, \pm} = \alpha_s(k_1^{-1}) + \alpha_s(k_2) \pm \alpha_s(k_2 J).$$

acting on $C^\omega((-1, \infty); V)$. We let

$$\text{SEF}_s^{\omega, \pm} := \{ \varphi \in C^\omega((-1, \infty); V) \mid \varphi = \mathcal{L}_s^{\text{slow}, \pm} \varphi \}$$

be the space of real-analytic eigenfunctions with eigenvalue 1 of $\mathcal{L}_s^{\text{slow}, \pm}$, and we let $\text{SEF}_s^{\omega, \text{as}, \pm}$ be the subspace of functions $\varphi \in \text{SEF}_s^{\omega, \pm}$ such that there exist $c_1, c_2 \in V$ (depending on φ) such that

$$\varphi(x) = c_1 x^{1-2s} + O_{x \rightarrow \infty}(x^{-2s}) \quad \text{and} \quad \varphi(x) = \frac{c_2}{x+1} + O_{x \rightarrow -1^+}(1).$$

Further we define $\text{SEF}_s^{\omega, \text{dec}, \pm}$ to be the subspace which consists of the functions $\varphi \in \text{SEF}_s^{\omega, \pm}$ for which the map

$$\begin{cases} \varphi \pm \alpha_s(Q)\varphi & \text{on } (0, \infty) \\ -\alpha_s(S)\varphi \mp \alpha_s(J)\varphi & \text{on } (-\infty, 0) \end{cases}$$

extends smoothly to \mathbb{R} , and the map

$$\begin{cases} \varphi & \text{on } (-1, \infty) \\ \mp \alpha_s(T^{-1}J)\varphi & \text{on } (-\infty, -1) \end{cases}$$

extends smoothly to $P^1(\mathbb{R})$.

Proposition 3.13 ([15]). *If $\text{Re } s \in (0, 1)$ and χ is the trivial representation then $\text{SEF}_s^{\omega, \text{dec}, \pm}$ is isomorphic (as a vector space) to the space of even resp. odd Maass cusp forms for Γ .*

In order to state the even and odd fast transfer operators for Γ let

$$E_a := (-1, 0), \quad E_b := (0, 1), \quad E_c := (1, \infty).$$

Further, for $\text{Re } s > \frac{1}{2}$, we set

$$\mathcal{L}_{1,s}^{\text{fast}} := \sum_{n \in \mathbb{N}} \alpha_s(k_1^{-n}), \quad \mathcal{L}_{2,s}^{\text{fast}} := \sum_{n \in \mathbb{N}} \alpha_s(k_2^n).$$

Then, for $\text{Re } s > \frac{1}{2}$, the even resp. odd fast transfer operator is

$$\mathcal{L}_s^{\text{fast}, \pm} = \begin{pmatrix} 0 & \pm \alpha_s(k_2 J) & \mathcal{L}_{1,s}^{\text{fast}} \\ \mathcal{L}_{2,s}^{\text{fast}} & \pm \alpha_s(k_2 J) & \mathcal{L}_{1,s}^{\text{fast}} \\ \mathcal{L}_{2,s}^{\text{fast}} & \pm \alpha_s(k_2 J) & 0 \end{pmatrix};$$

it acts on the projective Banach space

$$\mathcal{B} := \mathcal{B}(E_a) \oplus \mathcal{B}(E_b) \oplus \mathcal{B}(E_c).$$

For $\text{Re } s \leq \frac{1}{2}$, these transfer operators and their components are given by meromorphic continuation.

Let

$$\text{FEF}_s^{\pm} := \{f \in \mathcal{B} \mid f = \mathcal{L}_s^{\text{fast}, \pm} f\}$$

and let $\text{FEF}_s^{\text{dec}, \pm}$ be its subspace of functions $f = (f_a, f_b, f_c)^{\top} \in \text{FEF}_s^{\pm}$ such that

$$\begin{cases} f_b \pm \alpha_s(Q)(1 + \mathcal{L}_{1,s}^{\text{fast}})f_c & \text{on } (0, 1) \\ -\alpha_s(S)(1 + \mathcal{L}_{1,s}^{\text{fast}})f_c \mp \alpha_s(J)f_b & \text{on } (-1, 0) \end{cases}$$

extends smoothly to $(-1, 1)$,

$$\begin{cases} (1 + \mathcal{L}_{2,s}^{\text{fast}})f_a & \text{on } (-1, 0) \\ \mp \alpha_s(T^{-1}J)f_b & \text{on } (-2, -1) \end{cases}$$

extends smoothly to $(-2, 0)$, and

$$\begin{cases} \alpha_s(S) (1 + \mathcal{L}_{1,s}^{\text{fast}}) f_c & \text{on } (-1, 0) \\ \mp \alpha_s(ST^{-1}J) (1 + \mathcal{L}_{1,s}^{\text{fast}}) f_c & \text{on } (0, 1) \end{cases}$$

extends smoothly to $(-1, 1)$.

The proof of the following theorem is analogous to that of Theorem 3.4.

Theorem 3.14. *Let $s \in \mathbb{C} \setminus \{\frac{1}{2}\}$ with $\text{Re } s > 0$. Then the spaces $\text{SEF}_s^{\omega, \text{as}, \pm}$ and FEF_s^{\pm} are isomorphic as vector spaces. The isomorphism is given by*

$$\text{FEF}_s^{\pm} \rightarrow \text{SEF}_s^{\omega, \text{as}, \pm}, \quad f = (f_a, f_b, f_c)^{\top} \mapsto \varphi,$$

where

$$\varphi|_{E_a} := (1 + \mathcal{L}_{2,s}^{\text{fast}}) f_a, \quad \varphi|_{E_b} := f_b \quad \text{and} \quad \varphi|_{E_c} := (1 + \mathcal{L}_{1,s}^{\text{fast}}) f_c.$$

The converse isomorphism is

$$\text{SEF}_s^{\omega, \text{as}, \pm} \rightarrow \text{FEF}_s^{\pm}, \quad \varphi \mapsto f = (f_a, f_b, f_c)^{\top},$$

where f is determined by

$$f_a := ((1 - \alpha_s(k_2))\varphi)|_{E_a}, \quad f_b := \varphi|_{E_b} \quad \text{and} \quad f_c := ((1 - \alpha_s(k_1^{-1}))\varphi)|_{E_c}.$$

These isomorphisms induce isomorphisms between $\text{SEF}_s^{\omega, \text{dec}, \pm}$ and $\text{FEF}_s^{\text{dec}, \pm}$.

3.3. Isomorphism for non-cofinite Hecke triangle groups. Let

$$\Gamma := \Gamma_{\ell}$$

be a Hecke triangle group with parameter $\ell > 2$, thus a non-cofinite Fuchsian group. We consider the slow and fast transfer operators from [20, 15]. To improve readability we omit the dependence on ℓ in the notation.

Let

$$a_1 := \begin{bmatrix} 1 & \ell \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad a_2 := \begin{bmatrix} \ell & 1 \\ -1 & 0 \end{bmatrix}.$$

The even resp. odd slow transfer operator for Γ is

$$\mathcal{L}_s^{\text{slow}, \pm} = \alpha_s(a_2) + \alpha_s(a_1^{-1}) \pm \alpha_s(a_2 J),$$

acting on $C^{\omega}((-1, \infty); V)$. We define

$$\text{SEF}_s^{\omega, \pm} := \{ \varphi \in C^{\omega}((-1, \infty); V) \mid \varphi = \mathcal{L}_s^{\text{slow}, \pm} \varphi \}$$

to be the space of real-analytic eigenfunctions with eigenvalue 1 of $\mathcal{L}_s^{\text{slow}, \pm}$, and let

$$\text{SEF}_s^{\omega, \text{as}, \pm} := \{ \varphi \in \text{SEF}_s^{\omega, \pm} \mid \exists c \in V: \varphi(x) = cx^{1-2s} + O_{x \rightarrow \infty}(x^{-2s}) \}.$$

In order to state the fast even resp. odd transfer operator we set

$$E_1 := (-1, 1) \quad \text{and} \quad E_2 := (\ell - 1, \infty).$$

For $\text{Re } s > \frac{1}{2}$ we define

$$\mathcal{L}_{1,s}^{\text{fast}} := \sum_{n \in \mathbb{N}} \alpha_s(a_1^{-n}).$$

Then the fast even resp. odd transfer operator is (for $\operatorname{Re} s > \frac{1}{2}$)

$$\mathcal{L}_s^{\text{fast}, \pm} = \begin{pmatrix} \alpha_s(a_2) \pm \alpha_s(a_2 J) & \mathcal{L}_{1,s}^{\text{fast}} \\ \alpha_s(a_2) \pm \alpha_s(a_2 J) & 0 \end{pmatrix},$$

which acts on the projective Banach space

$$\mathcal{B} := \mathcal{B}(E_1) \oplus \mathcal{B}(E_2).$$

For $\operatorname{Re} s \leq \frac{1}{2}$, these transfer operators and their components are defined by meromorphic continuation. Let

$$\operatorname{FEF}_s^\pm := \{f \in \mathcal{B} \mid f = \mathcal{L}_s^{\text{fast}, \pm} f\}.$$

The proof of the following theorem is analogous to that of Theorem 3.4.

Theorem 3.15. *Let $s \in \mathbb{C} \setminus \{\frac{1}{2}\}$ with $\operatorname{Re} s > 0$. Then the spaces $\operatorname{SEF}_s^{\omega, \text{as}, \pm}$ and FEF_s^\pm are isomorphic as vector spaces. The isomorphism is given by*

$$\operatorname{FEF}_s^\pm \rightarrow \operatorname{SEF}_s^{\omega, \text{as}, \pm}, \quad f = (f_1, f_2)^\top \mapsto \varphi,$$

where

$$\varphi|_{(-1,1)} := f_1 \quad \text{and} \quad \varphi|_{(-1+\ell, \infty)} := (1 + \mathcal{L}_{1,s}^{\text{fast}}) f_2.$$

The inverse isomorphism is

$$\operatorname{SEF}_s^{\omega, \text{as}, \pm} \rightarrow \operatorname{FEF}_s^\pm, \quad \varphi \mapsto f = (f_1, f_2)^\top,$$

where f is determined by

$$f_1 := \varphi|_{(-1,1)} \quad \text{and} \quad f_2 := ((1 - \alpha_s(a_1^{-1}))\varphi)|_{(-1+\ell, \infty)}.$$

4. A FEW REMARKS

- (1) The isomorphism maps in Theorems 3.4-3.15 enjoy a certain structure that shows that Theorem A can be extended to a wider class of Fuchsian groups as soon as the details for the necessary transfer operators are provided. The existence of suitable pairs of slow/fast transfer operators is highly expected [15]. Moreover, if the considered Fuchsian group and the fast/slow transfer operators commute with some exterior symmetry (such as Q or J) then it is reasonable to expect that also a strengthening of Theorem A (in analogy to Theorem B) can be proven.

The principal objects for the isomorphism are the *slow* discretizations for the geodesic flow and the *slow* transfer operators. The *fast* discretizations and the *fast* transfer operators arise as follows: Whenever the acting element in the slow discretization is parabolic, one induces on this element in order to construct the fast discretization. More precisely, suppose that $p \in \operatorname{PSL}_2(\mathbb{R})$ is parabolic with fixed point $a \in \mathbb{R} \cup \{\infty\}$ and suppose further that the slow discrete dynamical system contains a component ('submap') of the form

$$(25) \quad (p^{-1}.b, a) \rightarrow (b, a), \quad x \mapsto p.x$$

(or $(a, p^{-1}.b) \rightarrow (a, b)$, $x \mapsto p.x$). Then, for the fast discretization, this submap is substituted by the maps ($n \in \mathbb{N}$)

$$(26) \quad (p^{-n}.b, p^{-(n+1)}.b) \rightarrow (b, p^{-1}.b), \quad x \mapsto p^n.x.$$

Let 1_W denote the characteristic function of any set W . The map in (25) contributes to the slow transfer operator the term

$$(27) \quad 1_{(b,a)} \cdot \alpha_s(p),$$

the map in (26) contributes to the fast transfer operator the term

$$(28) \quad 1_{(b,p^{-1}.b)} \cdot \sum_{n \in \mathbb{N}} \alpha_s(p^n).$$

In the previous sections we have only provided the (equivalent) matrix representations for transfer operators. We refer to [13] how to switch between those and (27)-(28).

At those places where the acting element is hyperbolic, the slow and the fast discretizations are identical. The guiding idea for the isomorphism map is that the eigenfunctions of the slow transfer operator and those of the fast transfer operator are ‘essentially identical’. Let f denote an eigenfunction with eigenvalue 1 of $\mathcal{L}_s^{\text{fast}}$, and φ an eigenfunction with eigenvalue 1 of $\mathcal{L}_s^{\text{slow}}$. Thus, at those intervals where the discretizations are identical, say at I_0 , f and φ should coincide:

$$f|_{I_0} = \varphi|_{I_0}.$$

Whenever a parabolic element acts in the submap then the effect of the induction/acceleration procedure needs to be inverted, which is done as follows for $\text{Re } s > \frac{1}{2}$: Let $I_p = (b, p^{-1}.b)$ and note that

$$f|_{I_p} = (1 - \alpha_s(p))\varphi|_{I_p}$$

yields

$$\sum_{n \in \mathbb{N}} \alpha_s(p^n) f = \alpha_s(p) \varphi$$

whenever $\varphi \in o(x^{-2s})$. Conversely, the formal inverse of $(1 - \alpha_s(p))$ is

$$\sum_{n=0}^{\infty} \alpha_s(p^n) = 1 + \sum_{n \in \mathbb{N}} \alpha_s(p^n).$$

Hence,

$$\varphi|_{I_p} = \left(1 + \sum_{n \in \mathbb{N}} \alpha_s(p^n)\right) f|_{I_p}.$$

For the consideration of all $\text{Re } s \leq \frac{1}{2}$, the meromorphic continuations for the operators are to be considered. The asymptotic expansion of the Lerch zeta function respectively asymptotic expansions of φ of the form as in Proposition 3.10 yield the necessary and sufficient growth properties of φ . In [1] we show the details for another pair of slow/fast transfer operators.

- (2) The explicit formulas for the isomorphism maps clearly show that additional conditions on eigenfunctions can be accommodated at least when they can be expressed in terms of acting elements.
- (3) Patterson conjectured a relation between the divisor of Selberg zeta functions and certain cohomology spaces [14] (see also [4, 7, 10]). For Fuchsian lattices Γ , Bruggeman, Lewis and Zagier provided a characterization of the space of Maass cusp forms for Γ with spectral parameter s as the space of parabolic 1-cohomology with values in the semi-analytic, smooth vectors of the principal

series representation for the parameter s [3]. In [13, 21, 15] the second author (for Γ_ℓ with $\ell < 2$ jointly with Möller) established an (explicit) isomorphism between $\text{SEF}_s^{\omega, \text{dec}, \pm}$ and the corresponding cohomology space from [3]. In turn, Theorems A and B support Patterson’s conjecture within a transfer operator framework.

We stress that the relation between these spectral singularities (namely, the spectral parameters of Maass cusp forms) of the Selberg zeta function and the (dimension of the) cohomology spaces which arises from these transfer operator techniques is canonical. In particular, it does not depend on the choice of an admissible discretization for the geodesic flow.

It would be interesting to see if for the singularities of the Selberg zeta function that do not arise from Maass cusp forms also such a cohomological interpretation of $\text{SEF}_s^{\omega, \text{as}, \pm}$ is possible. Moreover, it would be desirable to find an extension of such a cohomological framework which allows to include non-trivial representations as well as non-cofinite Fuchsian groups.

- (4) Further it would be desirable to characterize the elements in $\text{SEF}_s^{\omega, \text{as}, \pm}$ that are not contained in $\text{SEF}_s^{\omega, \text{dec}, \pm}$ purely in a transfer operator framework (in particular, without relying on Selberg theory). A complete characterization would allow us to provide a classification of the zeros of the Selberg zeta function that does not use the Selberg trace formula. For the case that Γ is the modular group $\text{PSL}_2(\mathbb{Z})$ and χ is the trivial representation, the combination of [12, 6, 5, 2, 8] provides such characterizations.

REFERENCES

- [1] A. Adam and A. Pohl, *Period functions and Selberg zeta functions for $\Gamma_0(2)$* , in preparation.
- [2] R. Bruggeman, *Automorphic forms, hyperfunction cohomology, and period functions*, J. reine angew. Math. **492** (1997), 1–39.
- [3] R. Bruggeman, J. Lewis, and D. Zagier, *Period functions for Maass wave forms. II: cohomology*, Mem. Am. Math. Soc. **237** (2015).
- [4] U. Bunke and M. Olbrich, *Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group*, Ann. Math. (2) **149** (1999), no. 2, 627–689.
- [5] C.-H. Chang and D. Mayer, *The period function of the nonholomorphic Eisenstein series for $\text{PSL}(2, \mathbb{Z})$* , Math. Phys. Electron. J. **4** (1998), Paper 6, 8.
- [6] ———, *The transfer operator approach to Selberg’s zeta function and modular and Maass wave forms for $\text{PSL}(2, \mathbb{Z})$* , Emerging applications of number theory (Minneapolis, MN, 1996), IMA Vol. Math. Appl., vol. 109, Springer, New York, 1999, pp. 73–141.
- [7] A. Deitmar and J. Hilgert, *Cohomology of arithmetic groups with infinite dimensional coefficient spaces*, Doc. Math. **10** (2005), 199–216 (electronic).
- [8] ———, *A Lewis correspondence for submodular groups*, Forum Math. **19** (2007), no. 6, 1075–1099.
- [9] J. Hilgert and A. Pohl, *Symbolic dynamics for the geodesic flow on locally symmetric orbifolds of rank one*, Proceedings of the fourth German-Japanese symposium on infinite dimensional harmonic analysis IV. On the interplay between representation theory, random matrices, special functions, and probability, Tokyo, Japan, September 10–14, 2007, Hackensack, NJ: World Scientific, 2009, pp. 97–111.
- [10] A. Juhl, *Cohomological theory of dynamical zeta functions*, Basel: Birkhäuser, 2001.
- [11] M. Katsurada, *Power series and asymptotic series associated with the Lerch zeta-function*, Proc. Japan Acad., Ser. A **74** (1998), no. 10, 167–170.
- [12] J. Lewis and D. Zagier, *Period functions for Maass wave forms. I*, Ann. of Math. (2) **153** (2001), no. 1, 191–258.

- [13] M. Möller and A. Pohl, *Period functions for Hecke triangle groups, and the Selberg zeta function as a Fredholm determinant*, Ergodic Theory Dynam. Systems **33** (2013), no. 1, 247–283.
- [14] S. Patterson, *On Ruelle’s zeta-function.*, Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday. Pt. II: Papers in analysis, number theory and automorphic L-functions, Pap. Workshop L-Funct., Number Theory, Harmonic Anal., Tel-Aviv/Isr. 1989, Isr. Math. Conf. Proc. 3, 163–184 (1990)., 1990.
- [15] A. Pohl, *Symbolic dynamics, automorphic functions, and Selberg zeta functions with unitary representations*, arXiv:1503.00525, to appear in Contemp. Math.
- [16] ———, *Symbolic dynamics for the geodesic flow on locally symmetric good orbifolds of rank one*, 2009, dissertation thesis, University of Paderborn, <http://d-nb.info/gnd/137984863>.
- [17] ———, *A dynamical approach to Maass cusp forms*, J. Mod. Dyn. **6** (2012), no. 4, 563–596.
- [18] ———, *Period functions for Maass cusp forms for $\Gamma_0(p)$: A transfer operator approach*, Int. Math. Res. Not. **14** (2013), 3250–3273.
- [19] ———, *Symbolic dynamics for the geodesic flow on two-dimensional hyperbolic good orbifolds*, Discrete Contin. Dyn. Syst., Ser. A **34** (2014), no. 5, 2173–2241.
- [20] ———, *A thermodynamic formalism approach to the Selberg zeta function for Hecke triangle surfaces of infinite area*, Commun. Math. Phys. **337** (2015), no. 1, 103–126.
- [21] ———, *Odd and even Maass cusp forms for Hecke triangle groups, and the billiard flow*, Ergodic Theory Dynam. Systems **36** (2016), no. 1, 142–172.
- [22] A. Pohl and V. Spratte, *A geometric reduction theory for indefinite binary quadratic forms over $\mathbb{Z}[\lambda]$* , arXiv:1512.08090.

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